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ORIGINAL PAPER

Optimal moment conditions for complete convergence for maximal normed weighted sums from arrays of rowwise independent random elements in Banach spaces

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Abstract
This work establishes complete convergence for maximal normed weighted sums from arrays of rowwise independent random elements taking values in a real separable stable type p Banach space or in a real separable Rademacher type p Banach space under optimal moment conditions. An extension of a result in Hu et al. (Stochas Anal Appl 39:177–193, 2021) is obtained as a special case of the main theorem. To establish the main result, which is a Baum–Katz–Hsu–Robbins–Erdös-type theorem for maximal normed weighted sums, we prove a Rosenthal-type inequality for maximal normed partial sums of independent random elements taking values in Rademacher type p Banach spaces. Moreover, the conditions for complete convergence in the main result are shown to completely characterize stable type p Banach spaces when $1 \leq p < 2$. The sharpness of the results is illustrated by two examples.

Keywords Maximal normed weighted sum · Complete convergence · Optimal moment condition · Rademacher type p Banach space · Stable type p Banach space · Rosenthal inequality

Mathematics Subject Classification 60F15 · 60B11 · 60B12

1 Introduction and motivation

Throughout, all random elements are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$.

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Let Λ be a nonempty index set. A family of $\mathcal{X}$-valued random elements $\{X_\lambda, \lambda \in \Lambda\}$ is said to be *stochastically dominated* by a real-valued random variable X if

$$\sup_{\lambda\in\Lambda} \mathbb{P}(\|X_\lambda\| > t) \le \mathbb{P}(|X| > t) \text{ for all } t \ge 0. \tag{1.1}$$

If the random elements $X_\lambda, \lambda \in \Lambda$ are identically distributed, then (1.1) is of course satisfied with $X = \|X_{\lambda_0}\|$ for any $\lambda_0 \in \Lambda$. Some authors use an apparently weaker definition of $\{X_\lambda, \lambda \in \Lambda\}$ being stochastically dominated by a real-valued random variable Y, namely that

$$\sup_{\lambda\in\Lambda} \mathbb{P}(\|X_\lambda\| > t) \le C_1\mathbb{P}(C_2|Y| > t) \text{ for all } t \ge 0 \tag{1.2}$$

for some constants $C_1, C_2 \in (0, \infty)$. It was recently shown by Rosalsky and Thành [21] that (1.1) and (1.2) are indeed equivalent.

If a family of random elements $\{X_\lambda, \lambda \in \Lambda\}$ is stochastically dominated by a real-valued random variable X, then for all $p > 0$ and $t > 0$,

$$\sup_{\lambda\in\Lambda} \mathbb{E}(\|X_\lambda\|^p \mathbf{1}(\|X_\lambda\| > t)) \le \mathbb{E}(|X|^p \mathbf{1}(|X| > t)) \tag{1.3}$$

and

$$\sup_{\lambda\in\Lambda} \mathbb{E}(\|X_\lambda\|^p \mathbf{1}(\|X_\lambda\| \le t)) \le \mathbb{E}(|X|^p \mathbf{1}(|X| \le t)) + t^p\mathbb{P}(|X| > t) \le \mathbb{E}(|X|^p). \tag{1.4}$$

The inequality in (1.3) follows from Lemma 3 of Adler et al. [2] and the first inequality in (1.4) follows from Lemma 1 of Adler and Rosalsky [1]. It is easy to verify the second inequality in (1.4). We will use (1.3) and (1.4) in our proofs without further mention.

Complete convergence for sums of independent $\mathcal{X}$-valued random elements was studied by many authors, but only few of them consider complete convergence for maximal normed partial sums which is of special interest. The following proposition is a recent result of Hu et al. [12] concerning complete convergence for maximal normed partial sums of random elements in Rademacher type p Banach spaces. We refer to Ledoux and Talagrand [16] for the definitions of Rademacher type p and stable type p Banach spaces. Equivalent characterizations of a Banach space being of stable type p, properties of stable type p Banach spaces, as well as various relationships between the conditions "Rademacher type p" and "stable type p" may be found in Woyczyński [28], Marcus and Woyczyński [17], and Pisier [18]. Some of these properties and relationships were listed in [20].

Proposition 1.1 (Hu et al. [12]) *Let $1 \le p \le 2$, and $\{k_n, n \ge 1\}$ be a sequence of positive integers satisfying $\lim_{n\to\infty} k_n = \infty$. Let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent mean zero random elements taking values in a real separable Rademacher type p Banach space. Suppose that the array $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is stochastically dominated by a real-valued random variable X. Let $\alpha > 0$, and let $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{k_n} |a_{n,i}|^p = O(n^{-\alpha}). \tag{1.5}$$

Suppose that

$$k_n = o(n^{\alpha/p}), \tag{1.6}$$

and

$$\mathbb{E}(|X|^p) < \infty. \tag{1.7}$$

Then for all $\beta < \alpha - 1$,

$$\sum_{n=1}^{\infty} n^\beta \mathbb{P}\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k} a_{n,i}X_{n,i}\right\| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.8}$$

In view of the Baum–Katz theorem (see Theorem 1 in [4]), a natural question to ask is whether or not the moment condition (1.7) is sharp. The current work is an attempt to answer this question. More precisely, we shall prove in Sect. 3 the following theorem which follows from Theorem 2.3, the main result of this paper.

Theorem 1.2 *Let* $1 \le p \le 2$, *and* $\{k_n, n \ge 1\}$ *be a sequence of positive integers satisfying* $\lim_{n\to\infty} k_n = \infty$. *Let* $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *be an array of rowwise independent mean zero random elements taking values in a real separable Rademacher type p Banach space. Suppose that the array* $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *is stochastically dominated by a real-valued random variable* X. *Let* $\alpha > 0$, *and let* $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *be an array of constants satisfying* (1.5). *Suppose that*

$$k_n = O(n). \tag{1.9}$$

Then for all $-1 \le \beta < \alpha - 1$, *the moment condition*

$$\mathbb{E}\left(|X|^{(\beta+2)p/(\alpha+1)}\right) < \infty \tag{1.10}$$

implies (1.8).

Remark 1.3 (i) It is easy to see that (1.8) is trivial if $\beta < -1$. Since $\beta < \alpha - 1$, the moment condition (1.10) is weaker than (1.7). Furthermore, it follows from Proposition 2.8 and from a perusal of the proof of Theorem 1.2 that the moment condition (1.10) is optimal for (1.8).

(ii) In Proposition 1.1, there is a tradeoff involving α. The large is α, the stronger is (1.5) but the weaker is (1.6). It is easy to see that if $\alpha > p$, then (1.6) is weaker than (1.9). But if $\alpha \le p$, then (1.6) is stronger than (1.9). See Remark 4.3 in [12] for another tradeoff involving p and the hypotheses of Proposition 1.1.

2 A Baum–Katz–Hsu–Robbins–Erdös-type theorem for maximal normed weighted sums

2.1 A Rosenthal-type inequality for maximal normed partial sums in Rademacher type *p* Banach spaces

The following result, which may be of independent interest, is a Rosenthal-type inequality for maximal normed partial sums of independent random elements in Rademacher type p Banach spaces. When $q = p$, it reduces to Lemma 2.1 of Rosalsky and Thành [19]. Making use of the case $q > p$ is a crucial step in the proof of Theorem 2.3.

Lemma 2.1 *Let $1 \le p \le 2$ and $\{X_i, 1 \le i \le n\}$ be a collection of n independent mean zero random elements in a Rademacher type p Banach space $\mathcal{X}$. Then for all $q \ge p$, there exists a constant $C_{p,q} \in (0,\infty)$ depending only on p and q such that*

$$\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\|^q\right) \le C_{p,q}\left(\left(\sum_{i=1}^{n}\mathbb{E}\|X_i\|^p\right)^{q/p} + \sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^q\right)\right). \tag{2.1}$$

Proof The conclusion of the lemma is trivial if $q = p = 1$. Therefore, we only need to consider the case $q > 1$. Let

$$S_k = X_1 + \cdots + X_k,\ 1 \le k \le n,$$

and

$$\mathcal{F}_k = \sigma(X_1, \ldots, X_k),\ 1 \le k \le n.$$

Then $\{\|S_k\|, \mathcal{F}_k, 1 \le k \le n\}$ is a nonnegative submartingale and so by Doob's inequality (see, e.g., [7, p. 255]),

$$\mathbb{E}\left(\max_{1\le k\le n}\|S_k\|^q\right) \le \left(\frac{q}{q-1}\right)^q \mathbb{E}\|S_n\|^q. \tag{2.2}$$

Since $\mathcal{X}$ is of Rademacher type p,

$$\mathbb{E}\left(\|S_n\|^p\right) \le C_p \sum_{i=1}^{k}\mathbb{E}\left(\|X_i\|^p\right), \tag{2.3}$$

where $C_p \in (0,\infty)$ is a constant depending only on p. Applying Lemma A.1, Jensen's inequality, and (2.3), there exists a constant C_q depending only on q such that for all $1 \le k \le n$,

$$\begin{aligned}
\mathbb{E}\left(\|S_n\|^q\right) &\le C_q\left((\mathbb{E}\|S_n\|)^q + \mathbb{E}\left(\max_{1\le i\le n}\|X_i\|\right)^q\right)\\
&\le C_q\left(\left(\mathbb{E}\,\|S_n\|^p\right)^{q/p} + \mathbb{E}\left(\sum_{i=1}^{n}\|X_i\|^q\right)\right)\\
&\le C_q\left(\left(C_p\sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^p\right)\right)^{q/p} + \sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^q\right)\right).
\end{aligned} \tag{2.4}$$

Combining (2.2) and (2.4), we have

$$\begin{aligned}
\mathbb{E}\left(\max_{1\le k\le n}\|S_k\|^q\right) &\le \left(\frac{q}{q-1}\right)^q C_q\left(\left(C_p\sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^p\right)\right)^{q/p} + \sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^q\right)\right)\\
&\le C_{p,q}\left(\left(\sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^p\right)\right)^{q/p} + \sum_{i=1}^{n}\mathbb{E}\left(\|X_i\|^q\right)\right)
\end{aligned}$$

thereby proving (2.1) with $C_{p,q} = \left(\frac{q}{q-1}\right)^q C_q \max\{C_p, 1\}$. □

Remark 2.2 The Rosenthal inequality (Rosenthal [22]) for sums of independent random variables is a very useful tool in proving limit theorem in probability. Finding the best constant in the inequality for independent random variables taking values in the real line (Rademacher type 2) is an interesting problem. If $\{X_i, 1 \le i \le n\}$ are independent symmetric real-valued random variables, it is proved in [13] that

$$\left(\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^q\right)^{1/q} \le \frac{Kq}{\log q} \max\left\{\left(\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^2\right)^{1/2}, \left(\sum_{i=1}^{n} \mathbb{E}(|X_i|^q)\right)^{1/q}\right\} \text{ for all } q \ge 2, \tag{2.5}$$

where K is a universal constant satisfying $\frac{1}{e\sqrt{2}} \le K \le 7.35$. In [13], it is also proved that the rate $q/\log q$ is optimal. In [15], it is shown that (2.5) holds with K approximately equal to $2e$ (see Theorem 2 and Corollary 3 in [15]). Recently, Chen et al. [6] used Stein's method and proved that (2.5) holds with $K \le 3.5$ without assuming the symmetry of the random variables. The Rosenthal-type inequality obtained in [6] can be applied to random variables satisfying many interesting dependence structures.

2.2 A Baum–Katz–Hsu–Robbins–Erdös-type theorem for maximal normed weighted sums

Theorem 2.3 may now be presented. It is a Baum–Katz–Hsu–Robbins–Erdös-type theorem (see Theorem 1 in [4], Theorem 1 in [10], and Theorem I in [8]) for maximal normed weighted sums in stable type p Banach spaces. Hereafter, let C denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Theorem 2.3 *Let $p \ge 1$, $\alpha > 1/2$, $p_0 = \min\{p, 2\}$. Let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent mean zero random elements in a real separable stable type p_0 Banach space $\mathcal{X}$, and let $\{k_n, n \ge 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$ and (1.9) holds. Suppose that the array $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is stochastically dominated by a real-valued random variable X. If*

$$\mathbb{E}(|X|^p) < \infty, \tag{2.6}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k} c_{n,i}X_{n,i}\right\| > n^{\alpha}\varepsilon\right) < \infty \textit{ for all } \varepsilon > 0, \tag{2.7}$$

where $\{c_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is an array of constants satisfying

$$\sum_{i=1}^{k_n} |c_{n,i}|^r = O(n) \textit{ for some } r > \max\{p, 2(\alpha p - 1)/(2\alpha - 1)\}. \tag{2.8}$$

Remark 2.4 By (1.9) and Hölder's inequality, we see that the larger is r, the stronger is the condition (2.8). To see this, assume that

$$\sum_{i=1}^{k_n} |c_{n,i}|^{r_1} = O(n) \text{ for some } r_1 > 0.$$

Then for all $0 < r_2 < r_1$, we have from Hölder's inequality that

$$\sum_{i=1}^{k_n} |c_{n,i}|^{r_2} \le \left(\sum_{i=1}^{k_n} 1\right)^{1-r_2/r_1} \left(\sum_{i=1}^{k_n} |c_{n,i}|^{r_1}\right)^{r_2/r_1} = O(n).$$

Proof of Theorem 2.3 It is clear to see that we only need to consider the case $\alpha p \ge 1$ since (2.7) is obvious if $\alpha p < 1$. In view of (1.9), we can assume, without loss of generality, that $k_n = n$. For $n \ge 1$, $1 \le k \le n$, set

$$Y_{n,k} = X_{n,k}\mathbf{1}(\|X_{n,k}\| \le n^\alpha),\ Z_{n,k} = X_{n,k}\mathbf{1}(\|X_{n,k}\| > n^\alpha),$$

and

$$S_{n,k} = \sum_{i=1}^{k} c_{n,i}(Y_{n,i} - \mathbb{E}(Y_{n,i})).$$

Let $\varepsilon > 0$ be arbitrary. For $n \ge 1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}X_{n,i}\right\| > n^\alpha\varepsilon\right)\\
&\le \mathbb{P}\left(\max_{1\le i\le n}\|X_{n,i}\| > n^\alpha\right) + \mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}Y_{n,i}\right\| > n^\alpha\varepsilon\right) \qquad (2.9)\\
&\le \sum_{i=1}^{n}\mathbb{P}\left(\|X_{n,i}\| > n^\alpha\right) + \mathbb{P}\left(\max_{1\le k\le n}\left\|S_{n,k}\right\| > n^\alpha\varepsilon - \sum_{i=1}^{n}\left\|\mathbb{E}\left(c_{n,i}Y_{n,i}\right)\right\|\right).
\end{aligned}$$

By (2.6), Lemma A.2, and the stochastic domination assumption, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{P}\left(\|X_{n,i}\| > n^\alpha\right) \le \sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{P}\left(|X| > n^\alpha\right) < \infty. \qquad (2.10)$$

Since $\mathbb{E}(X_{n,i}) = 0$, $\mathbb{E}(|X|^p) < \infty$ and $\alpha p \ge 1$, we have by (2.8), Remark 2.4, and the Lebesgue dominated convergence theorem that

$$\begin{aligned}
\frac{\sum_{i=1}^{n}\|\mathbb{E}\left(c_{n,i}Y_{n,i}\right)\|}{n^\alpha} &= \frac{\sum_{i=1}^{n}\|\mathbb{E}\left(c_{n,i}Z_{n,i}\right)\|}{n^\alpha}\\
&\le \frac{\left(\sum_{i=1}^{n}|c_{n,i}|\right)\mathbb{E}(|X|\mathbf{1}(|X| > n^\alpha))}{n^\alpha}\\
&\le \frac{n\mathbb{E}(|X|\mathbf{1}(|X| > n^\alpha))}{n^\alpha}\\
&= \frac{1}{n^{\alpha-1}}\mathbb{E}\left(\frac{|X|^p}{|X|^{p-1}}\mathbf{1}(|X| > n^\alpha)\right)\\
&\le \frac{1}{n^{\alpha p-1}}\mathbb{E}(|X|^p\mathbf{1}(|X|^p > n^{\alpha p})) \to 0 \text{ as } n \to \infty. \qquad (2.11)
\end{aligned}$$

Combining (2.9)–(2.11), the proof of the theorem will be completed if we can show that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left\|S_{n,k}\right\| > n^\alpha\varepsilon/2\right) < \infty. \qquad (2.12)$$

Firstly, we consider the case where $1 \le p < 2$. Then $p_0 = p$. Since $\mathcal{X}$ is of stable type p Banach space, it is of Rademacher type p_1 for some $p < p_1 < 2$. Set $q = \min\{p_1, r\} \in (p, r]$. By (2.8) and Remark 2.4, we have

$$\sum_{i=1}^{n} |c_{n,i}|^q \le O(n). \tag{2.13}$$

Since $\mathcal{X}$ is of Rademacher type p_1, it is also of Rademacher type q. Applying Markov's inequality, Lemma 2.1, (2.13), (2.6), and Lemma A.2, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n} \|S_{n,k}\| > n^{\alpha}\varepsilon/2\right) \\
&\quad\le \frac{2^q}{\varepsilon^q}\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\mathbb{E}\left(\max_{1\le k\le n} \|S_{n,k}\|^q\right) \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\sum_{i=1}^{n}|c_{n,i}|^q\mathbb{E}\|Y_{n,i}-\mathbb{E}(Y_{n,i})\|^q \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\sum_{i=1}^{n}|c_{n,i}|^q\mathbb{E}\|Y_{n,i}\|^q \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\left(\sum_{i=1}^{n}|c_{n,i}|^q\right)\left(\mathbb{E}(|X|^q\mathbf{1}(|X|\le n^{\alpha}))+n^{\alpha q}\mathbb{P}(|X|>n^{\alpha})\right) \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-1}\mathbb{E}(|X|^q\mathbf{1}(|X|\le n^{\alpha}))+C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{P}(|X|>n^{\alpha})<\infty
\end{aligned} \tag{2.14}$$

thereby proving (2.12).

We now consider the case where $p \ge 2$. Then $p_0 = 2$ and $\mathcal{X}$ is of Rademacher type 2. Applying Markov's inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n} \|S_{n,k}\| > n^{\alpha}\varepsilon/2\right) &\le \frac{2^r}{\varepsilon^r}\sum_{n=1}^{\infty} n^{\alpha(p-r)-2}\mathbb{E}\left(\max_{1\le k\le n} \|S_{n,k}\|^r\right) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha(p-r)-2}\left(\left(\sum_{i=1}^{n} c_{n,i}^2\mathbb{E}\|Y_{n,i}-\mathbb{E}(Y_{n,i})\|^2\right)^{r/2}\right. \\
&\quad\left.+\sum_{i=1}^{n}|c_{n,i}|^r\mathbb{E}\|Y_{n,i}-\mathbb{E}(Y_{n,i})\|^r\right) \\
&:= I_1 + I_2.
\end{aligned} \tag{2.15}$$

By Jensen's inequality and (2.6), we have

$$\begin{aligned}
\mathbb{E}\|Y_{n,i}\|^2 &\le \mathbb{E}(|X|^2\mathbf{1}(|X|\le n^{\alpha}))+n^{2\alpha}\mathbb{P}(|X|>n^{\alpha}) \\
&\le \mathbb{E}(X^2) \le (\mathbb{E}|X|^p)^{2/p} < \infty.
\end{aligned} \tag{2.16}$$

It follows from (2.8), (2.16), and Remark 2.4 that

$$
\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty} n^{\alpha(p-r)-2}\left(\sum_{i=1}^{n} c_{n,i}^2\right)^{r/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha(p-r)-2} n^{r/2} \\
&= C\sum_{n=1}^{\infty} n^{-1+(\alpha p-1)-r(2\alpha-1)/2} < \infty. \qquad (2.17)
\end{aligned}
$$

Proceeding in a similar manner as the last four lines of (2.14), we obtain that

$$I_2 < \infty. \qquad (2.18)$$

Combining (2.15), (2.17), and (2.18), we obtain (2.12).

The proof of the theorem is completed. □

Remark 2.5 If $0 < p < 1$, then Theorem 2.3 holds without any geometric condition of the underlying Banach space and without the rowwise independence and mean zero assumptions on the array $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$. This is proved as follows:

Let $Y_{n,k}$ and $Z_{n,k}$ be as in the proof of Theorem 2.3. Let $\varepsilon > 0$ be arbitrary, and let $q = \min\{(p+1)/2, r\} \in (p, 1)$. By (2.8) and Remark 2.4, we have

$$\sum_{i=1}^{n} |c_{n,i}|^q = O(n). \qquad (2.19)$$

Applying Markov's inequality, the C_r-inequality (see, e.g., Theorem 3.2.2 in [9]), (2.19), (2.6), and Lemma A.2, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}Y_{n,i}\right\| > n^{\alpha}\varepsilon/2\right) \\
&\quad\le \frac{2^q}{\varepsilon^q}\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\mathbb{E}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}Y_{n,i}\right\|^q\right) \\
&\quad\le \frac{2^q}{\varepsilon^q}\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\mathbb{E}\left(\left(\sum_{i=1}^{n} \|c_{n,i}Y_{n,i}\|\right)^q\right) \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\sum_{i=1}^{n} |c_{n,i}|^q\mathbb{E}\|Y_{n,i}\|^q \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-2}\left(\sum_{i=1}^{n} |c_{n,i}|^q\right)\left(\mathbb{E}(|X|^q\mathbf{1}(|X|\le n^{\alpha})) + n^{\alpha q}\mathbb{P}(|X| > n^{\alpha})\right) \\
&\quad\le C\sum_{n=1}^{\infty} n^{\alpha(p-q)-1}\mathbb{E}(|X|^q\mathbf{1}(|X|\le n^{\alpha})) + C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{P}(|X| > n^{\alpha}) < \infty. \qquad (2.20)
\end{aligned}
$$

Combining the first inequality in (2.9), (2.10), and (2.20), we obtain (2.7). □

For the case where the Banach space is of Rademacher type 2 (or, equivalently, of stable type 2), we have the following corollary which establishes complete convergence for maximal normed weighted sums. When $a_{n,i} \equiv 1$, this result was proved in [11] (only for partial sums of real-valued random variables) by a different method (see [11, Theorem 2]). When $s = 1$, this is the Hsu–Robbins theorem (see Theorem 1 in [10]) for weighted sums in Banach spaces.

Corollary 2.6 *Let $1 \leq s < 2$ and $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent mean zero random elements in a real separable Rademacher type 2 Banach space. Assume that the array $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is stochastically dominated by a real-valued random variable X. If*

$$\mathbb{E}(|X|^{2s}) < \infty, \tag{2.21}$$

then

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} c_{n,i} X_{n,i} \right\| > n^{1/s}\varepsilon \right) < \infty \text{ for all } \varepsilon > 0, \tag{2.22}$$

where $\{c_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying

$$\sum_{i=1}^{n} |c_{n,i}|^r = O(n) \text{ for some } r > 2s/(2-s). \tag{2.23}$$

Proof Letting $p = 2s$ and $\alpha = 1/s$, then (2.21) coincides with (2.6), and (2.8) follows from (2.23) and the assumption that $1 \leq s < 2$. Applying Theorem 2.3 with $p = 2s, \alpha = 1/s$ and $p_0 = 2$, we obtain (2.22). □

The following example, which was inspired by examples of Beck [5] and Kuczmaszewska and Szynal [14], shows that Theorem 2.3 can fail if the stable type p_0 hypothesis is weakened to the Rademacher type p_0 hypothesis.

Example 2.7 Let $1 \leq p < 2$ and $\alpha = 1/p$. Then $\alpha > 1/2$. Let $\mathcal{X}$ be the real separable Banach space ℓ_p of absolute p-th power summable real sequences $x = (x_1, x_2, \ldots)$ with norm

$$\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}.$$

Let $p_0 = \min\{p, 2\} = p$. The Banach space ℓ_p is of Rademacher type p_0 but is not of stable type p_0. Let e_i denote the i-th element of the standard basis in ℓ_p; that is, e_i is the element in ℓ_p having 1 for its i-th coordinate and 0 for the other coordinates. Let $k_n = n,\ n \geq 1$. Let $\{R_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent Rademacher random variables:

$$\mathbb{P}(R_{n,i} = 1) = \mathbb{P}(R_{n,i} = -1) = \frac{1}{2},\ 1 \leq i \leq n, n \geq 1.$$

Define

$$X_{n,i} = R_{n,i} e_i,\ 1 \leq i \leq n, n \geq 1.$$

Clearly (1.9) holds and the array is stochastically dominated by the random variable $X \equiv 1$ statisfying (2.6) since

$$\|X_{n,i}\| \equiv 1 \text{ a.s.},\ 1 \leq i \leq n, n \geq 1.$$

Now (2.8) holds with $c_{n,i} \equiv 1,\ 1 \le i \le n, n \ge 1$ for all $r > 0$. However

$$\left\| \sum_{i=1}^{k} c_{n,i} X_{n,i} \right\| = \left\| \sum_{i=1}^{k} R_{n,i} e_i \right\| = \left(\sum_{i=1}^{k} 1 \right)^{1/p} = k^{1/p} \text{ a.s.},\ 1 \le k \le n, n \ge 1$$

and so for all $\varepsilon \in (0, 1)$,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P} \left(\max_{1 \le k \le k_n} \left\| \sum_{i=1}^{k} c_{n,i} X_{n,i} \right\| > n^{\alpha} \varepsilon \right) &= \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left(\left\| \sum_{i=1}^{n} c_{n,i} X_{n,i} \right\| > n^{1/p} \varepsilon \right) \\ &= \sum_{n=1}^{\infty} n^{-1} = \infty. \end{aligned}$$

Thus (2.7) fails.

The following proposition shows that the moment condition (2.6) in Theorem 2.3 is optimal for (2.7).

Proposition 2.8 *Let $p \ge 1,\ p_0 = \min\{p, 2\}$. Let $\{X_i, i \ge 1\}$ be a sequence of independent mean zero random elements in a real separable stable type p_0 Banach space such that the random variables $\|X_n\|$, $n \ge 1$ are identically distributed. Then the following two statements are equivalent.*

(i) The random variable $\|X_1\|$ satisfies

$$\mathbb{E}(\|X_1\|^p) < \infty.$$

(ii) For every $\alpha > 1/2$ and for every array of constants $\{c_{n,i}, 1 \le i \le n, n \ge 1\}$ satisfying (2.8), we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P} \left(\max_{1 \le k \le n} \left\| \sum_{i=1}^{k} c_{n,i} X_i \right\| > n^{\alpha} \varepsilon \right) < \infty \text{ for all } \varepsilon > 0. \tag{2.24}$$

Proof The proof of implication ((i)⇒(ii)) follows immediately from Theorem 2.3 by letting $X_{n,i} = X_i$ for all $n \ge 1$ and $1 \le i \le n$, and letting $X = \|X_1\|$.

Next, assume that (ii) holds. Letting $\alpha = 1$ and $c_{n,i} \equiv 1$, we have from (2.24) that

$$\sum_{n=1}^{\infty} n^{p-2} \mathbb{P} \left(\max_{1 \le k \le n} \left\| \sum_{i=1}^{k} X_i \right\| > n\varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

The rest of the argument proceeds in exactly the same manner as that of the necessary part of Theorem 1.2 of [3] or that of Theorem 3.1 (ii) of [27]. We note that the assumption $p < 2$ in the necessary part of Theorem 1.2 of [3] or Theorem 3.1 (ii) of [27] is not needed. □

The following example shows that in a real separable stable type $p = 2$ Banach space,

(a) Theorem 2.3 can fail if $\alpha = 1/2$,

and

(b) the implication ((i)⇒(ii)) in Proposition 2.8 can fail if $\alpha = 1/2$.

Example 2.9 Consider the real separable stable type $p = 2$ Banach space $\mathcal{X} = \mathbb{R}$ with norm $|x|,\ x \in \mathbb{R}$. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. real-valued random variables with $\mathbb{E}(X_1) = 0$ and $0 < \mathbb{E}(X_1^2) < \infty$, and let $X_{n,i} = X_i$, $1 \leq i \leq n$, $n \geq 1$. Let $\alpha = 1/2$, and $c_{n,i} = 1$, $1 \leq i \leq n$, $n \geq 1$. Now, if Theorem 2.3 holds with $\alpha = 1/2$ or if the implication ((i)$\Rightarrow$(ii)) in Proposition 2.8 holds with $\alpha = 1/2$, then

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| > n^{1/2}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0,$$

and thus

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/2}} = 0 \ \text{ a.s.} \tag{2.25}$$

But it follows from the central limit theorem that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/2}} = \infty \ \text{ a.s.}$$

which contradicts (2.25) thereby verifying (a) and (b).

The next theorem shows that when $1 \leq p < 2$, Theorem 2.3 provides an exact characterization of stable type p Banach spaces.

Theorem 2.10 *Let $1 \leq p < 2$, and let $\mathcal{X}$ be a real separable Banach space. Then the following two statements are equivalent.*

(i) $\mathcal{X}$ is of stable type p.
(ii) For every sequence $\{X_n, n \geq 1\}$ of independent mean zero $\mathcal{X}$-valued random elements which is stochastically dominated by a random variable X, for every $\alpha > 1/2$, and for every array of constants $\{c_{n,i}, 1 \leq i \leq n, n \geq 1\}$ satisfying (2.8), the condition $\mathbb{E}(|X|^p) < \infty$ implies

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} c_{n,i} X_i\right\| > n^{\alpha}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.26}$$

Proof The proof of implication ((i)$\Rightarrow$(ii)) follows immediately from Theorem 2.3 by letting $X_{n,i} = X_i$ for all $n \geq 1$ and $1 \leq i \leq n$.

Next, assume that (ii) holds. Let $\{X_n, n \geq 1\}$ be a sequence of independent and symmetric $\mathcal{X}$-valued random elements which is stochastically dominated by a real-valued random variable X with $\mathbb{E}(|X|^p) < \infty$. By letting $\alpha = 1/p > 1/2$ and $c_{n,i} \equiv 1$, (2.26) becomes

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\| > n^{1/p}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0,$$

and therefore

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} = 0 \text{ a.s.}$$

By Lemma A.3, $\mathcal{X}$ is of stable type p. ◻

Remark 2.11 A Reviewer kindly suggested that in Theorem 2.3, it may be possible to replace (2.8) by the weaker condition that

$$\sum_{i=1}^{k_n} |c_{n,i}|^r = O(n) \text{ for some } r > p. \tag{2.27}$$

That Reviewer also kindly pointed out to us that in Sung [23] and Wu et al. [29], the authors proved Theorem 2.3 for *sequences* (rather than for *arrays*) of real-valued identically distributed ρ^*-mixing random variables under (2.27). It should be noted that if $1 \le p \le 2$ or $\sup_{n\ge1, 1\le i\le k_n} |c_{n,i}| \le c_0 < \infty$ for some positive constant c_0, then (2.8) and (2.27) are the same. However, (2.27) is weaker than (2.8) if $p > 2$ and the $c_{n,i}$ are not uniformly bounded.

By using our proof and further employing the technique presented in Sung [23] and Wu et al. [29], we can prove that Theorem 2.3 (and so Proposition 2.8 and Theorem 2.10) still holds if (2.8) is weakened by (2.27). We sketch the proof as follows.

Under the assumptions of Theorem 2.3 with (2.8) replaced by (2.27), we can assume, without loss of generality, that $k_n \equiv n$ and $\sum_{i=1}^n |c_{n,i}|^r \le n$ for all $n \ge 1$. Since (2.8) and (2.27) are the same if $1 \le p \le 2$, we only need to consider the case where $p > 2$, and therefore, $\alpha > 1/p$. Set

$$c_{n,i}^{(1)} = c_{n,i}\mathbf{1}(|c_{n,i}| \le 1),\ c_{n,i}^{(2)} = c_{n,i}\mathbf{1}(|c_{n,i}| > 1),\ 1 \le i \le n, n \ge 1.$$

To prove (2.7), it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}^{(1)} X_{n,i}\right\| > n^\alpha\varepsilon\right) < \infty \text{ for all } \varepsilon > 0, \tag{2.28}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} c_{n,i}^{(2)} X_{n,i}\right\| > n^\alpha\varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.29}$$

Since $\sup_{n\ge1,1\le i\le n} |c_{n,i}^{(1)}| \le 1$, we obtain (2.28) by Theorem 2.3. Therefore, it remains to prove (2.29). Let $\varepsilon > 0$ be arbitrary. For $1 \le k \le n$, $n \ge 1$, set $Y_{n,k}^{(2)} = c_{n,k}^{(2)} X_{n,k}\mathbf{1}(\|c_{n,k}^{(2)} X_{n,k}\| \le n^\alpha)$ and $S_{n,k}^{(2)} = \sum_{i=1}^k (Y_{n,i}^{(2)} - \mathbb{E}(Y_{n,i}^{(2)}))$. By (2.15)–(2.17) in Sung [23], we have

$$I_1 := \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{P}(|c_{n,i}^{(2)} X| > n^\alpha) \le C\mathbb{E}|X|^p < \infty. \tag{2.30}$$

By (2.21)–(2.23) in Sung [23] (see also Lemma 2.7 in Wu et al. [29]), we have for all $s > r$ that

$$I_2 := \sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2}\sum_{i=1}^{n}\mathbb{E}(|c_{n,i}^{(2)} X|^s\mathbf{1}(|c_{n,i}^{(2)} X| \le n^\alpha)) \le C\mathbb{E}|X|^p < \infty. \tag{2.31}$$

By using (2.30) and following the same argument which leads to (2.12), the proof of (2.29) will be complete if we can show that

$$J := \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left\|S_{n,k}^{(2)}\right\| > n^\alpha\varepsilon/2\right) < \infty. \tag{2.32}$$

Letting $s > 2(\alpha p - 1)/(2\alpha - 1)$, and using (2.30), (2.31) and the same argument as (2.15), we have

$$
\begin{aligned}
J &\le C \sum_{n=1}^{\infty} n^{\alpha(p-s)-2} \left(\left(\sum_{i=1}^{n} \mathbb{E}\|Y_{n,i}^{(2)} - \mathbb{E}(Y_{n,i}^{(2)})\|^2 \right)^{s/2} + \sum_{i=1}^{n} \mathbb{E}\|Y_{n,i}^{(2)} - \mathbb{E}(Y_{n,i}^{(2)})\|^s \right) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha(p-s)-2} \left(\left(\sum_{i=1}^{n} \left(c_{n,i}^{(2)}\right)^2 \right)^{s/2} + \sum_{i=1}^{n} \mathbb{E}\|Y_{n,i}^{(2)}\|^s \right) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha(p-s)-2} \left(n^{s/2} + \sum_{i=1}^{n} \left(\mathbb{E}(|c_{n,i}^{(2)} X|^s \mathbf{1}(|c_{n,i}^{(2)} X| \le n^{\alpha})) + n^{s\alpha} \mathbb{P}(|c_{n,i}^{(2)} X| > n^{\alpha}) \right) \right) \\
&\le C \left(\sum_{n=1}^{\infty} n^{-1+(\alpha p-1)-s(2\alpha-1)/2} + I_1 + I_2 \right) < \infty
\end{aligned}
$$

verifying (2.32). The proof is completed. □

3 The proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Proof of Theorem 1.2 Set

$$p_1 = (\beta + 2)p/(1 + \alpha), \ \alpha_1 = (1 + \alpha)/p.$$

Then $p_1 < p$ and $\alpha_1 p_1 = \beta + 2 \ge 1$. Since the underlying Banach space is of Rademacher type p, it is of stable type p_1. Letting $r_1 = p > p_1$ and $c_{n,i} \equiv n^{\alpha_1} a_{n,i}$, we have from (1.5) that

$$\sum_{i=1}^{k_n} |c_{n,i}|^{r_1} = n^{1+\alpha} \sum_{i=1}^{k_n} |a_{n,i}|^p = O(n).$$

Now with p replaced by p_1, α replaced by α_1, and r replaced by r_1, we obtain by applying Theorem 2.3 for the case $p_1 \ge 1$ and Remark 2.5 for the case $p_1 < 1$ that

$$\sum_{n=1}^{\infty} n^{\alpha_1 p_1 - 2} \mathbb{P} \left(\max_{1 \le k \le k_n} \left\| \sum_{i=1}^{k} c_{n,i} X_{n,i} \right\| > n^{\alpha_1} \varepsilon \right) < \infty \text{ for all } \varepsilon > 0,$$

thereby proving (1.8). □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Appendix

In this section, we present some known results which are used in the previous sections. The first lemma is Theorem 1 in [24].

Lemma A.1 *Let* $\{X_i, 1 \le i \le n\}$ *be a collection of* n *independent mean* 0 *random elements in a real separable Banach space* $\mathcal{X}$. *Then for all* $q \ge 1$, *there exists a constant* C_q *depending only on* q *such that*

$$\mathbb{E}\left(\left\|\sum_{i=1}^{n} X_i\right\|^q\right) \le C_q \left(\left(\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|\right)^q + \mathbb{E}\left(\max_{1\le i\le n} \|X_i\|^q\right)\right).$$

The following simple result can be obtained by standard estimate (see Theorem 2.12.3 in [9] or Lemma 4 in Thành [26] for a more general version of condition (ii)).

Lemma A.2 *Let* $\alpha > 0$, $q > p > 0$ *and let* X *be a real-valued random variable. Then the following three statements are equivalent.*

(i) $\mathbb{E}(|X|^p) < \infty$.
(ii) $\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{P}(|X| > n^\alpha) < \infty$.
(iii) $\sum_{n=1}^{\infty} n^{\alpha(p-q)-1}\mathbb{E}(|X|^q \mathbf{1}(|X| \le n^\alpha)) < \infty$.

The next lemma is Lemma 2.3 in [20]. This lemma may be compared with Lemma 2.4 of [25] or Theorem V.9.1 of [28].

Lemma A.3 *Let* $1 \le p < 2$ *and let* $\mathcal{X}$ *be a real separable Banach space. Suppose for every sequence* $\{X_n, n \ge 1\}$ *of independent and symmetric* $\mathcal{X}$*-valued random elements which is stochastically dominated by a real-valued random variable* X *with* $\mathbb{E}(|X|^p) < \infty$ *that*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} = 0 \text{ in probability.}$$

Then $\mathcal{X}$ *is of stable type* p.

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